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GOVERNING EQUATIONS AND COMPLETE MATHEMATICAL PROOFS FOR COULOMB'S THEORY OF EARTH PRESSURE

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Abstract

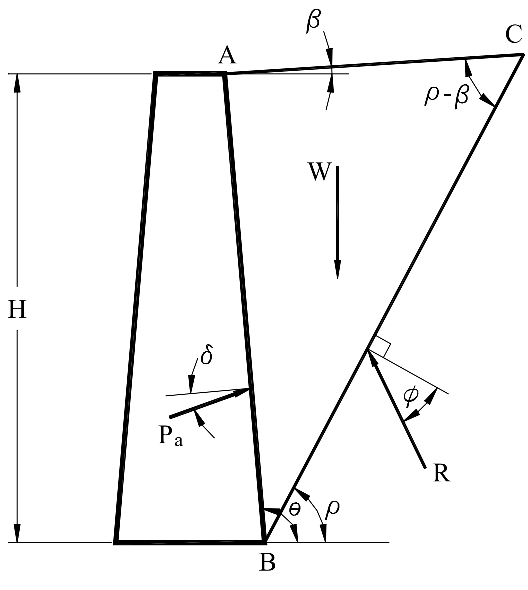
Coulomb's theory of earth pressure, proposed in 1776, has been widely used by engineers to design retaining walls for 245 years. However, because the earth pressure functions are very complicated, there is still no study providing governing equations and complete mathematical proofs. Therefore, this paper provides governing equations for solving Coulomb's theory of earth pressure using the first-order differential equations of Coulomb's earth pressure functions. Furthermore, the second-order differential equations of Coulomb's earth pressure functions are used to mathematically prove that the active earth pressure is the maximum value and the passive earth pressure is the minimum value. This not only completes the mathematical proofs for Coulomb's theory of earth pressures, but also enhances the physical understanding

and mathematical basis of the calculation of Coulomb's earth pressures.

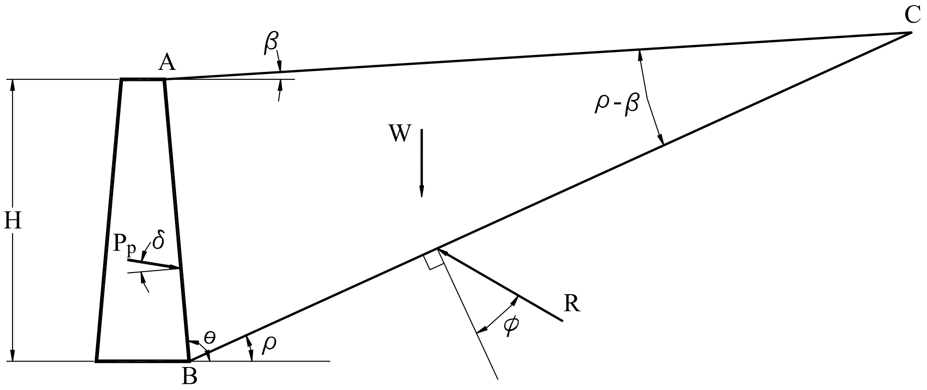
Keywords: Coulomb's theory, active and passive earth pressures, governing equation.

Introduction

In Figures 1(a) and (b), ΔABC is the potential active and the potential passive sliding failure block for a retaining wall, as proposed by Coulomb (1776). Here, H is the height of the retaining wall; γ is the unit weight of the soil; W is the weight per unit length of ΔABC; β is the inclination angle of $\overline{AC}$; θ is the inclination angle of $\overline{AB}$; ρ is the inclination angle of the potential active failure surface $\overline{BC}$; $\rho-\beta$ is the angle between $\overline{CA}$ and $\overline{CB}$; R is the resultant shear resisting force acting on $\overline{BC}$, the intersection angle of R and the normal line of $\overline{BC}$ is internal friction angle ϕ; and P_a is the active earth pressure acting on $\overline{AB}$, and the angle of intersection between P_a and the normal of $\overline{AB}$ is wall friction angle δ.



(a) Under active conditions



(b) Under passive conditions

Figure 1. Various forces on a retaining wall provided by Coulomb's potential sliding failure block.

When the inclination angles of the potential sliding failure surfaces corresponding to the active and passive earth pressure functions are changed such that P_a becomes a maximum value and P_p becomes a minimum value, P_a and P_p respectively become Coulomb's active and passive earth pressures (Coulomb, 1776).

Because Coulomb's earth pressure function (Coulomb, 1776) is too complicated, so far none of the current literatures (Bowles, 1988; Das, 2010; Fang, et al., 2003; Lambe and Whitman, 1969; McCarthy, 2007; Kramer, 1996; Pantelidis, 2019; Rebhann, 1987; Taylor, 1948; Terzaghi, 1944; US Army Corps of Engineers, 2019) providing complete mathematical proofs of active earth pressure P_a as the maximum value and passive earth pressure P_p as the minimum value. Therefore, in this paper, the authors will directly provide Coulomb's earth pressure function first-order differential equation equal to zero, and second-order differential equation less than and greater than zero, respectively, to provide complete mathematical proofs of Coulomb's active earth pressure P_a as the maximum value and passive earth pressure P_p as the minimum value.

Coulomb's Earth Pressure Functions

In Fig. 1 and Fig. 2, the potential sliding failure blocks under Coulomb's active and passive conditions (Coulomb, 1776), respectively, are given by ΔABC, where the weight W of ΔABC can be calculated as follows:

$$W = \frac{1}{2}\gamma \overline{BC} \cdot \overline{AB} \cdot \sin(\theta - \rho)$$
$$= \frac{1}{2}\gamma \overline{AB}^2 \cdot \frac{\sin(180^\circ - \theta + \beta)}{\sin(\rho - \beta)} \cdot \sin(\theta - \rho)$$
$$= \frac{1}{2}\gamma H^2 \frac{\sin(180^\circ - \theta + \beta)}{\sin(\rho - \beta)} \cdot \frac{\sin(\theta - \rho)}{\sin^2(180^\circ - \theta)}$$
$$= \frac{1}{2}\gamma H^2 \frac{\sin(\theta - \beta)}{\sin^2\theta} \cdot \frac{\sin(\theta - \rho)}{\sin(\rho - \beta)} \quad \text{(Equation 1)}$$

Coulomb's active earth pressure function

Figure 2 shows the force polygon of W, R, and P_a in Figure 1(a). Under the conditions of force balance, the force polygon is closed.

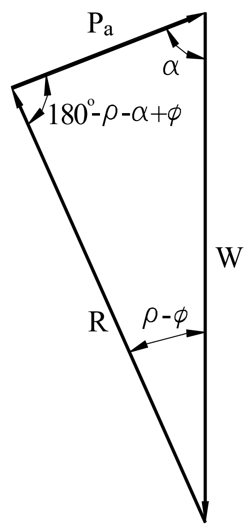


Figure 2. The closed force polygon of W, R, and P_a shown in Figure 1(a).

According to the law of sines, Coulomb's active earth pressure P_a shown in Figure 2 can be determined as follows:

$$P_a = W\frac{\sin(\rho-\phi)}{\sin(180^\circ-\rho-\alpha+\phi)} = \frac{1}{2}\gamma H^2\frac{\sin(\theta-\beta)}{\sin^2\theta}\cdot\frac{\sin(\theta-\rho)}{\sin(\rho-\beta)}\cdot\frac{\sin(\rho-\phi)}{\sin(\rho+\alpha-\phi)}. \qquad \text{(Equation 2)}$$

Coulomb's active earth pressure P_a can also be expressed as:

$$P_a = \frac{1}{2}\gamma H^2 K_a \qquad \text{(Equation 3)}$$

where K_a is the coefficient of active earth pressure. It is defined as:

$$K_a = \frac{\sin(\theta-\beta)}{\sin^2\theta}\cdot\frac{\sin(\theta-\rho)}{\sin(\rho-\beta)}\cdot\frac{\sin(\rho-\phi)}{\sin(\rho+\alpha-\phi)}. \qquad \text{(Equation 4)}$$

Coulomb's passive earth pressure function

Figure 3 shows the force polygon of W, R, and P_p in Figure 1(b).

Under the conditions of force balance, the force polygon is closed.

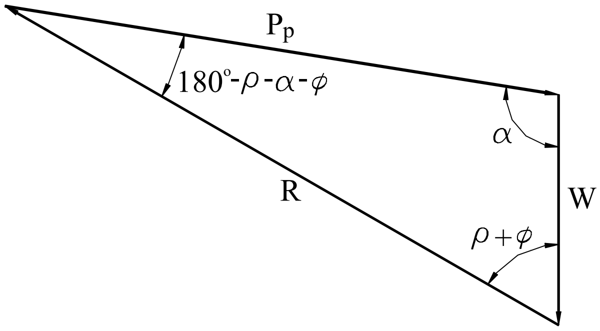


Figure 3. The closed force polygon of W, R, and P_p shown in Figure 1(b)

According to the law of sines, Coulomb's passive earth pressure P_p shown in Figure 3 can be obtained as follows:

$$P_p = W\frac{\sin(\rho+\phi)}{\sin(180^\circ-\rho-\alpha-\phi)}$$
$$=\frac{1}{2}\gamma H^2\frac{\sin(\theta-\beta)}{\sin^2\theta}\cdot\frac{\sin(\theta-\rho)}{\sin(\rho-\beta)}\cdot\frac{\sin(\rho+\phi)}{\sin(\rho+\alpha+\phi)}. \qquad \text{(Equation 5)}$$

Coulomb's passive earth pressure P_p can also be expressed as:

$$P_p = \frac{1}{2}\gamma H^2 K_p \qquad \text{(Equation 6)}$$

where K_p is the coefficient of passive earth pressure. It is defined as:

$$K_p = \frac{\sin(\theta-\beta)}{\sin^2\theta}\cdot\frac{\sin(\theta-\rho)}{\sin(\rho-\beta)}\cdot\frac{\sin(\rho+\phi)}{\sin(\rho+\alpha+\phi)}. \qquad \text{(Equation 7)}$$

Governing Equation For Coulomb's Earth Pressures

Governing equation for Coulomb's active earth pressure

For the case $\alpha = 180^\circ - \theta - \delta$, if and only if $\rho \neq \theta$, $\rho \neq \phi$, $\rho \neq \beta$, and $\rho \neq \phi - \alpha$, then the governing equation for Coulomb's active earth pressure is:

$$\cot(\theta - \rho) + \cot(\rho - \beta) = \cot(\rho - \phi) - \cot(\rho + \alpha - \phi). \qquad \text{(Equation 8)}$$

Governing equation for Coulomb's passive earth pressure

For the case $\alpha = 180^\circ - \theta + \delta$, if and only if $\rho \neq \theta$ and $\rho \neq \beta$, then the governing equation for Coulomb's passive earth pressure is:

$$\cot(\theta - \rho) + \cot(\rho - \beta) = \cot(\rho + \phi) - \cot(\rho + \alpha + \phi). \qquad \text{(Equation 9)}$$

Coulomb's Theory Of Earth Pressure

Coulomb's theory of active earth pressure

If and only if Coulomb's active earth pressure function shown in Equation 2 exists, then Equation 8 is the governing equation for solving Coulomb's active earth pressure P_a.

Coulomb's theory of passive earth pressure

If and only if Coulomb's passive earth pressure function shown in Equation 5 exists, then Equation 9 is the governing equation for solving Coulomb's passive earth pressure P_p.

Mathematical Proofs

Mathematical proof of Coulomb's active earth pressure

1) If H, γ, θ, β, ϕ, and δ are known, then $\frac{1}{2}\gamma H^2 \frac{\sin(\theta - \beta)}{\sin^2\theta}$ in Equation 2 is a constant C_a, and $\frac{\sin(\theta - \rho)}{\sin(\rho - \beta)} \cdot \frac{\sin(\rho - \phi)}{\sin(\rho + \alpha - \phi)}$ only

changes with the change of ρ. Thus, Coulomb's active earth pressure function can be rewritten as:

$$P_a = -C_a \cdot f_a(\rho).$$

In the above formula, since $-f_a(\rho) = \frac{\sin(\rho-\theta)}{\sin(\rho-\beta)} \cdot \frac{\sin(\rho-\phi)}{\sin(\rho+\alpha-\phi)}$, the extreme value of Coulomb's active earth pressure can be obtained by using the formula $\frac{dP_a}{d\rho} = C_a \cdot \frac{d[-f_a(\rho)]}{d\rho} = 0$. Then, the governing equation for solving Coulomb's active earth pressure P_a can be obtained as follows:

$$\begin{aligned}\frac{d[-f_a(\rho)]}{d\rho} &= \frac{\cos(\rho-\theta)}{\sin(\rho-\theta)}[-f_a(\rho)] + \frac{\cos(\rho-\phi)}{\sin(\rho-\phi)}[-f_a(\rho)] \\ &\quad - \frac{\cos(\rho-\beta)}{\sin(\rho-\beta)}[-f_a(\rho)] - \frac{\cos(\rho+\alpha-\phi)}{\sin(\rho+\alpha-\phi)}[-f_a(\rho)] \\ &= [\cot(\rho-\theta) + \cot(\rho-\phi) - \cot(\rho-\beta) - \cot(\rho+\alpha-\phi)] \cdot [-f_a(\rho)] \\ &= 0.\end{aligned}$$

For general retaining walls, since $\rho \neq \theta$, $\rho \neq \phi$, $\rho \neq \beta$, $\rho \neq \phi - \alpha$, and $-f_a(\rho)>0$ in the above formula, the governing equation of $\frac{dP_a}{d\rho} = 0$ is obtained as

$$\cot(\theta-\rho) + \cot(\rho-\beta)) = \cot(\rho-\phi) - \cot(\rho+\alpha-\phi).$$

When $\cot(\theta-\rho) + \cot(\rho-\beta) = \cot(\rho-\phi) - \cot(\rho+\alpha-\phi)$ is known, it is necessary to further prove that Coulomb's active earth pressure is the maximum value.

Since both C_a and $-f_a(\rho)$ are greater than zero, to prove that Coulomb's active earth pressure P_a is a maximum value, it is necessary to either

prove $\frac{d^2P_a}{d\rho^2} = C_a \cdot \frac{d^2[-f_a(\rho)]}{d\rho^2} < 0$ or

prove $\frac{d^2[-f_a(\rho)]}{d\rho^2} < 0$.

For general retaining walls, since $45^\circ < \theta < 135^\circ$, $|\beta| \leq \phi$, and $\delta \leq \phi$, the value of ρ that satisfies Equation 8 also satisfies $|\rho-\theta| < |\rho+\alpha-\phi|$ and

$|\rho-\phi|<|\rho-\beta|$. Therefore, $\csc^2(\rho+\alpha-\phi)<\csc^2(\rho-\theta)$ and $\csc^2(\rho-\beta)<\csc^2(\rho-\phi)$ both exist. Under such circumstances, the second-order differential equation of $-f_a(\rho)$ can be obtained as follows:

$$\frac{d^2[-f_a(\rho)]}{d\rho^2}=\left[-\csc^2(\rho-\theta)+\csc^2(\rho+\alpha-\phi)-\csc^2(\rho-\phi)+\csc^2(\rho-\beta)\right]\cdot[-f_a(\rho)]$$
$$+\left[\cot(\rho-\theta)-\cot(\rho+\alpha-\phi)+\cot(\rho-\phi)-\cot(\rho-\beta)\right]\cdot\frac{d[-f_a(\rho)]}{d\rho}.$$

Since $\cot(\theta-\rho)+\cot(\rho-\beta)=\cot(\rho-\phi)-\cot(\rho+\alpha-\phi)$, the above formula can be simplified to:

$$\frac{d^2[-f_a(\rho)]}{d\rho^2}=\left[-\csc^2(\rho-\theta)+\csc^2(\rho+\alpha-\phi)-\csc^2(\rho-\phi)+\csc^2(\rho-\beta)\right]\cdot[-f_a(\rho)]$$
$$<0.$$

This completes the mathematical proof of the *if* component "if Coulomb's active earth pressure function P_a shown in Equation 2 exists, then Equation 8 is the governing equation for solving Coulomb's active earth pressure P_a."

2) Since Equation 8 exists, the value of ρ that satisfies Equation 8 also makes $\sin(\rho-\theta)$, $\sin(\rho-\phi)$, $\sin(\rho-\beta)$, and $\sin(\rho+\alpha-\phi)$ non-zero. Therefore, after multiplying both sides of Equation 8 by $-f_a(\rho)$, the following formula can be obtained:

$$[\cot(\rho-\theta)+\cot(\rho-\phi)-\cot(\rho-\beta)-\cot(\rho+\alpha-\phi)]\cdot[-f_a(\rho)]$$
$$=\frac{\cos(\rho-\theta)}{\sin(\rho-\theta)}[-f_a(\rho)]+\frac{\cos(\rho-\phi)}{\sin(\rho-\phi)}[-f_a(\rho)]$$
$$-\frac{\cos(\rho-\beta)}{\sin(\rho-\beta)}[-f_a(\rho)]-\frac{\cos(\rho+\alpha-\phi)}{\sin(\rho+\alpha-\phi)}[-f_a(\rho)]$$
$$=\frac{d[-f_a(\rho)]}{d\rho}$$
$$=0.$$

For general retaining walls, since $45^\circ < \theta < 135^\circ$, $|\beta| \le \phi$, and $\delta \le \phi$, the constant C_a and $-f_a(\rho)$ are both greater than zero. Therefore, the value of ρ that satisfies Equation 8 also makes $|\rho-\theta| < |\rho+\alpha-\phi|$ and $|\rho-\phi| < |\rho-\beta|$. Therefore, $\csc^2(\rho-\theta) > \csc^2(\rho+\alpha-\phi)$ and $\csc^2(\rho-\phi) > \csc^2(\rho-\beta)$. Thus, the following relationship exists:

$$\left[-\csc^2(\rho-\theta)+\csc^2(\rho+\alpha-\phi)-\csc^2(\rho-\phi)+\csc^2(\rho-\beta)\right]\cdot[-f_a(\rho)]$$

$$=\left[-\csc^2(\rho-\theta)+\csc^2(\rho+\alpha-\phi)-\csc^2(\rho-\phi)+\csc^2(\rho-\beta)\right]\cdot[-f_a(\rho)]$$

$$+\left[\cot(\rho-\theta)-\cot(\rho+\alpha-\phi)+\cot(\rho-\phi)-\cot(\rho-\beta)\right]\cdot\frac{d[-f_a(\rho)]}{d\rho}$$

$$=\frac{d^2[-f_a(\rho)]}{d\rho^2}<0\,.$$

This completes the mathematical proof of the *only if* component: "only if the governing equation of Coulomb's active earth pressure P_a shown in Equation 8 exists, then P_a shown in Equation 2 is Coulomb's active earth pressure."

Mathematical proof of Coulomb's passive earth pressure

1) If H, γ, θ, β, ϕ, and δ are known, then $\frac{1}{2}\gamma H^2\frac{\sin(\theta-\beta)}{\sin^2\theta}$ in Equation 5 is a constant C_p, and $\frac{\sin(\rho-\theta)}{\sin(\rho-\beta)}\cdot\frac{\sin(\rho+\phi)}{\sin(\rho+\alpha+\phi)}$ only changes with the change of ρ. Thus, Coulomb's passive earth pressure function P_p can be rewritten as:

$$P_p=-C_p\cdot f_p(\rho)$$

In the above formula, since $-f_p(\rho)=\frac{\sin(\rho-\theta)}{\sin(\rho-\beta)}\cdot\frac{\sin(\rho+\phi)}{\sin(\rho+\alpha+\phi)}$, the extreme value of Coulomb's passive earth pressure can be obtained by using the formula $\frac{dP_p}{d\rho}=C_p\cdot\frac{d[-f_p(\rho)]}{d\rho}=0$. Then, the governing equation for solving Coulomb's passive earth pressure P_p can be obtained as follows:

$$\begin{aligned}\frac{d[-f_p(\rho)]}{d\rho}&=\frac{\cos(\rho-\theta)}{\sin(\rho-\theta)}[-f_p(\rho)]+\frac{\cos(\rho+\phi)}{\sin(\rho+\phi)}[-f_p(\rho)]\\&-\frac{\cos(\rho-\beta)}{\sin(\rho-\beta)}[-f_p(\rho)]-\frac{\cos(\rho+\alpha+\phi)}{\sin(\rho+\alpha+\phi)}[-f_p(\rho)]\\&=[\cot(\rho-\theta)+\cot(\rho+\phi)-\cot(\rho-\beta)-\cot(\rho+\alpha+\phi)]\cdot[-f_p(\rho)]\\&=0.\end{aligned}$$

For general retaining walls, since $\rho\neq\theta$ and $\rho\neq\beta$, $\sin(\rho-\theta)$, $\sin(\rho+\phi)$, $\sin(\rho-\beta)$, and $\sin(\rho+\alpha+\phi)$ are all non-zero, and $-f_p(\rho)>0$ in the above formula, the governing equation for $dP_p/d\rho=0$ is $\cot(\theta-\rho)+\cot(\rho-\beta)=\cot(\rho+\phi)-\cot(\rho+\alpha+\phi)$.

When $\cot(\theta-\rho)+\cot(\rho-\beta)=\cot(\rho+\phi)-\cot(\rho+\alpha+\phi)$ is known, it is necessary to further prove that Coulomb's passive earth pressure is the minimum value.

Since both C_p and $-f_p(\rho)$ are greater than zero, to prove that P_p is a minimum value, it is necessary to prove either $\frac{d^2P_p}{d\rho^2}=C_p\cdot\frac{d^2[-f_p(\rho)]}{d\rho^2}>0$ or $\frac{d^2[-f_p(\rho)]}{d\rho^2}>0$.

For general retaining walls, since $45^\circ<\theta<135^\circ$, $|\beta|\leq\phi$, and $\delta\leq\phi$, the value of ρ that satisfies Equation 9 also satisfies $|\rho-\theta|>|180^\circ-\rho-\alpha-\phi|$ and $|\rho+\phi|>|\rho-\beta|$. Therefore, $[-\csc^2(\rho-\theta)+\csc^2(\rho+\alpha+\phi)]>0$ and $[-\csc^2(\rho+\phi)+\csc^2(\rho-\beta)]>0$ both exist. Under such circumstances, the second-order differential formula of $-f_p(\rho)$ can be obtained as follows:

$$\frac{d^2[-f_p(\rho)]}{d\rho^2} = \left[-\csc^2(\rho-\theta)+\csc^2(\rho+\alpha+\phi)-\csc^2(\rho+\phi)+\csc^2(\rho-\beta)\right]\cdot\left[-f_p(\rho)\right]$$
$$+\left[\cot(\rho-\theta)-\cot(\rho+\alpha+\phi)+\cot(\rho+\phi)-\cot(\rho-\beta)\right]\cdot\frac{d\left[-f_p(\rho)\right]}{d\rho}.$$

In the above formula, since $\cot(\theta-\rho)$ $+\cot(\rho-\beta)=\cot(\rho+\phi)-\cot(\rho+\alpha+\phi)$, the following is obtained:

$$\frac{d^2[-f_p(\rho)]}{d\rho^2} = \left[-\csc^2(\rho-\theta)+\csc^2(\rho+\alpha+\phi)-\csc^2(\rho+\phi)+\csc^2(\rho-\beta)\right]\cdot\left[-f_p(\rho)\right]$$
$$>0.$$

This completes the mathematical proof of the *if* component: "If Coulomb's passive earth pressure function P_p shown in Equation 5 exists, then Equation 9 is the governing equation for solving Coulomb's passive earth pressure."

When Equation 9 exists, the value of ρ that satisfies Equation 9 also makes $\sin(\rho-\theta)$, $\sin(\rho+\phi)$, $\sin(\rho-\beta)$, and $\sin(\rho+\alpha+\phi)$ non-zero, and because $\sin(\rho+\phi)>0$ and $\sin(\rho+\alpha+\phi)>0$, the following formula can be obtained by multiplying both sides of $\cot(\rho-\theta)-\cot(\rho+\alpha+\phi)$ $+\cot(\rho+\phi)-\cot(\rho-\beta)=0$ by $-f_p(\rho)$.

$$\cot(\rho-\theta)[-f_p(\rho)]+\cot(\rho+\phi)[-f_p(\rho)]$$
$$-\cot(\rho-\beta)[-f_p(\rho)]-\cot(\rho+\alpha+\phi)[-f_p(\rho)]$$
$$=\frac{\cos(\rho-\theta)}{\sin(\rho-\theta)}[-f_p(\rho)]+\frac{\cos(\rho+\phi)}{\sin(\rho+\phi)}[-f_p(\rho)]$$
$$-\frac{\cos(\rho-\beta)}{\sin(\rho-\beta)}[-f_p(\rho)]-\frac{\cos(\rho+\alpha+\phi)}{\sin(\rho+\alpha+\phi)}[-f_p(\rho)]$$
$$=\frac{d[-f_p(\rho)]}{d\rho}$$
$$=0.$$

For general retaining walls, since $45^\circ < \theta < 135^\circ$, $|\beta| \le \phi$, and $\delta \le \phi$, the constant C_a and $-f_a(\rho)$ are both greater than zero. Thus, ρ satisfying Equation 9 also leads to $C_p \cdot \frac{d[-f_p(\rho)]}{d\rho} = \frac{dP_p}{d\rho} = 0$. Furthermore, because $|\rho - \theta| > |180^\circ - \rho - \alpha - \phi|$ and $|\rho + \phi| > |\rho - \beta|$ exist, it is known that $\csc^2(\rho - \theta) < \csc^2(\rho + \alpha + \phi)$ and $\csc^2(\rho + \phi) < \csc^2(\rho - \beta)$. Therefore, the following relationship can be obtained:

$$\frac{d^2[-f_p(\rho)]}{d\rho^2} = \left[-\csc^2(\rho-\theta) + \csc^2(\rho+\alpha+\phi) - \csc^2(\rho+\phi) + \csc^2(\rho-\beta)\right] \cdot \left[-f_p(\rho)\right]$$

$$+ \left[\cot(\rho-\theta) - \cot(\rho+\alpha+\phi) + \cot(\rho+\phi) - \cot(\rho-\beta)\right] \cdot \frac{d\left[-f_p(\rho)\right]}{d\rho}$$

$$= \left[-\csc^2(\rho-\theta) + \csc^2(\rho+\alpha+\phi) - \csc^2(\rho+\phi) + \csc^2(\rho-\beta)\right] \cdot \left[-f_p(\rho)\right]$$

$$> 0.$$

This completes the mathematical proof of the *only if* component: “Only if the governing equation of Coulomb’s active earth pressure P_p shown in Equation 9 exists, then P_p shown in Equation 5 is Coulomb’s passive earth pressure.”

Comparison And Discussion Of Results

1) Rebhann (1871) first obtained the relationship between Coulomb’s active earth pressure P_a and the weight W for the potential sliding failure block by using the closed force polygon of W, R, and P_a shown in Figure 3 and the law of sines. A drawing method (detailed in Figure 5) is used to illustrate that when the inclination angle ρ of the sliding failure surface changes such that $dP_a / d\rho = 0$, then the area of ΔABC equals the area of ΔBCE. Finally, Rebhann derived a formula for the coefficient of lateral earth pressure K_a under active conditions:

$$K_a = \frac{\sin^2(\theta-\phi)}{\sin^2\theta\cdot\sin(\theta+\delta)\left(1+\sqrt{\dfrac{\sin(\phi+\delta)\cdot\sin(\phi-\beta)}{\sin(\theta+\delta)\cdot\sin(\theta-\beta)}}\right)^2}. \qquad \text{(Equation 10)}$$

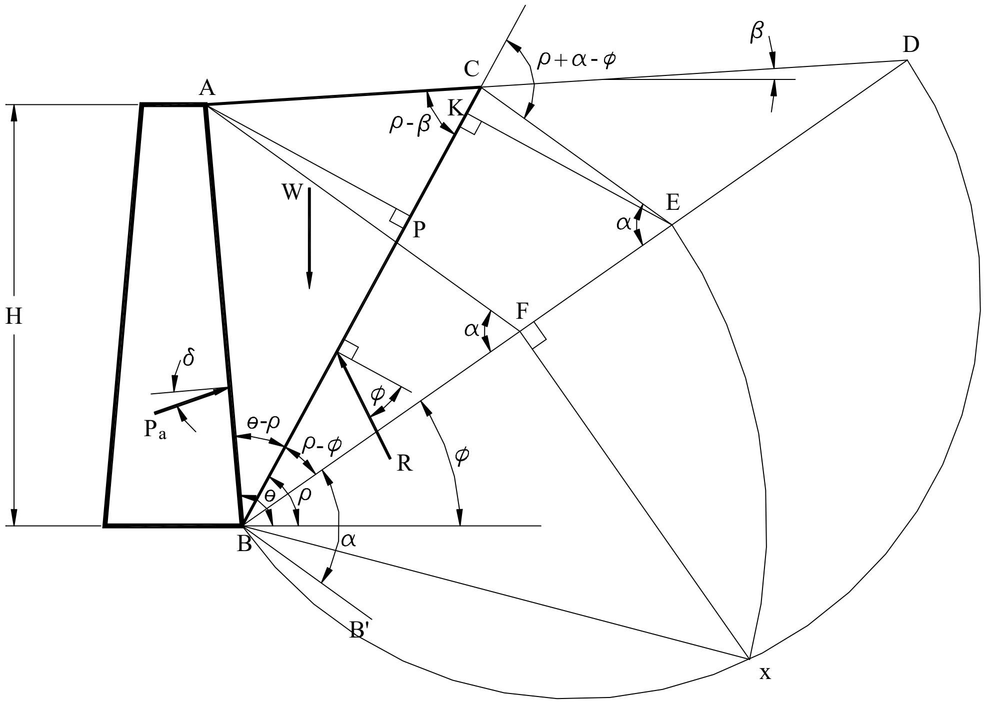


Figure 5. Rebhann's drawing method for Coulomb's active earth pressure.

2) Rebhann (1871) also obtained the relationship between Coulomb's passive earth pressure function P_p and the weight W of the potential sliding failure block by using the closed force polygon of the three forces of W, R, and P_p shown in Figure 4 and the law of sines. Then, a drawing method (detailed in Figure 6) was used to illustrate that when the inclination angle ρ of the sliding failure surface is changed such that $dP_a/d\rho=0$, then the area of ΔABC equals the area of ΔBCE. Finally, Rebhann derived a formula for the coefficient of lateral earth pressure K_a under passive conditions:

$$K_p = \frac{\sin^2(\theta+\phi)}{\sin^2\theta\cdot\sin(\theta-\delta)\left(1-\sqrt{\frac{\sin(\phi+\delta)\cdot\sin(\phi+\beta)}{\sin(\theta-\delta)\cdot\sin(\theta-\beta)}}\right)^2}. \quad \text{(Equation 11)}$$

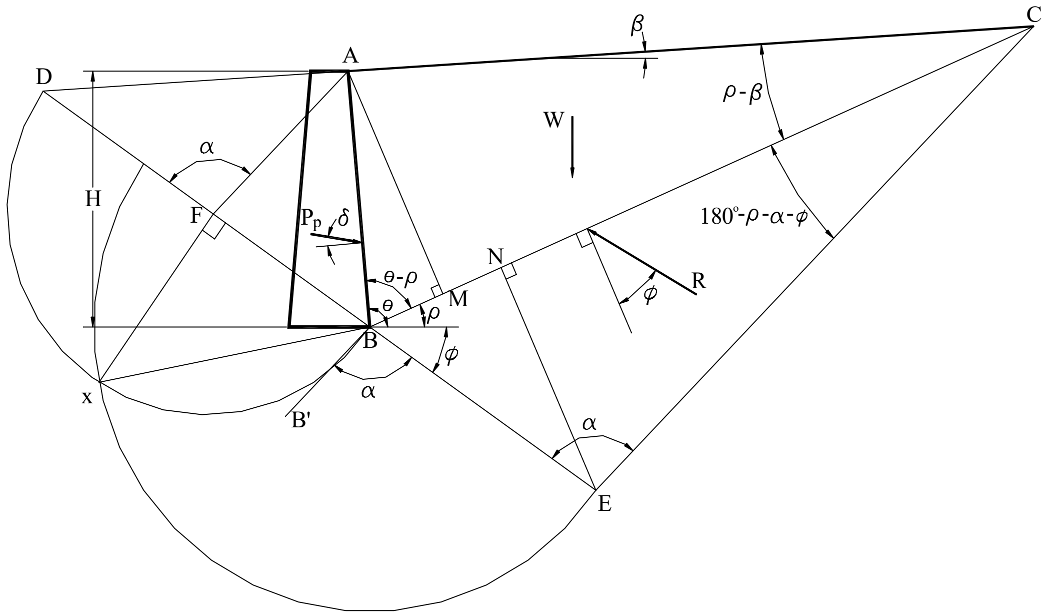


Figure 6. Rebhann's drawing method for Coulomb's passive earth pressure.

3) After obtaining the graphical relationships for $dP_a/d\rho = 0$ and $dP_p/d\rho = 0$ by the drawing method, Rebhann (1871) did not go on to prove that P_a is a maximum value by $d^2P_a/d\rho^2 < 0$ and that P_p is a minimum value by $d^2P_p/d\rho^2 > 0$. Therefore, the proof for the theory of Coulomb's earth pressure is incomplete.

4) When the inclination angles of the sliding failure surfaces corresponding to the maximum value of P_a and the minimum value of P_p are required, it is easy to solve them by using Equation 8 and Equation 9,

respectively. However, it is very time-consuming and laborious work to obtain them using Rebhann's drawing methods.

5) For the governing equation of Coulomb's active earth pressure, according to Figure 5, $\cot(\theta-\rho)=\overline{BP}/\overline{AP}$, $\cot(\rho-\beta)=\overline{CP}/\overline{AP}$, $\cot(\rho-\phi)=\overline{BK}/\overline{EK}$, and $\cot(\rho+\alpha-\phi)=-\overline{CK}/\overline{EK}$ can be obtained. If the above relational expressions are substituted into Eq. 8, then it can be found that:

$$\frac{\overline{BP}}{\overline{AP}}+\frac{\overline{CP}}{\overline{AP}}=\frac{\overline{BK}}{\overline{EK}}+\frac{\overline{CK}}{\overline{EK}}$$

The above formula can be simplified to:

$$\frac{\overline{BC}}{\overline{AP}}=\frac{\overline{BC}}{\overline{EK}}$$

Multiplying both sides by $\frac{1}{2}\overline{AP}\cdot\overline{EK}$ gives:

$$\frac{1}{2}\overline{BC}\cdot\overline{AP}=\frac{1}{2}\overline{BC}\cdot\overline{EK}$$

In Figure 5, this relationship means that the area of ΔABC equals the area of ΔBCE. Therefore, it is proved that the governing equation of Coulomb's active earth pressure proposed in this paper has the same physical meaning as the graphical relationship suitable for $dP_a/d\rho=0$ proposed by Rebhann (1871).

6) According to Figure 6, the same method can be used to prove that the physical meaning of the governing equation of Coulomb's passive earth pressure proposed in this paper is the

same as that of the graphical relationship suit able for $dP_p / d\rho = 0$ proposed by Rebhann (1871).

7) Take the retaining wall shown in Figures 1 or 2 as examples. When H=6 m, γ=20 kN/m^3, θ=120°, β=12°, c=0, ϕ=30°, and δ=20°, the following results can be obtained according to the traditional method and the method proposed in this paper:

(1) According to the traditional method, Coulomb's active and passive lateral earth pressure coefficients, *i.e.*, K_a=0.7882 and K_p=5.0000, are first obtained using Equations 10 and 11, respectively. Then, Coulomb's active and passive earth pressures, *i.e.*, P_a=283.752 kN per meter of wall and P_p=1800.000 kN per meter of wall, are determined using Equations 3 and 6, respectively. Regarding the inclination angles ρ of the sliding failure surfaces and the weight W of the sliding failure blocks for Coulomb's active and passive conditions, technicians rarely present these data because they require significant drawing efforts.

(2) According to the method proposed in this paper, by using the governing equations of Equations 8 and 9, the inclination angle (*i.e.*, $\rho = 57.57^{\circ}$) of the sliding failure surface for Coulomb's active conditions and the inclination angle (*i.e.*, $\rho = 40.00^{\circ}$) of the sliding failure surface for Coulomb's passive conditions can be obtained. Then, the known values of ρ can be substituted into Equation 1 to obtain the weight (*i.e.*, W = 566.679 kN per meter of wall) for the sliding failure block of Coulomb's active conditions and the weight (*i.e.*, W = 969.786 kN per meter of wall) for the sliding failure block of Coulomb's passive conditions. When the known values of ρ are substituted into Equations 2 and 5, this yields P_a = 283.752 kN per meter of wall and P_p = 1800.000 kN per meter of wall for the Coulomb's active and passive conditions,

respectively. Finally, when the known values of ρ are substituted into Equations 4 and 7, the Coulomb active earth pressure coefficient (*i.e.*, K_a=0.7882) and the Coulomb passive earth pressure coefficient (*i.e.*, K_p=5.0000) can be obtained.

Conclusions And Suggestions

Coulomb's theory of earth pressure has been widely used for 245 years in the design of retaining walls. However, because the earth pressure functions are very complicated, there have been no complete mathematical proofs. To complete Coulomb's theory of earth pressure, complete mathematical proofs were provided for the first time in this paper.

Through comparison, it is known that the proposed governing equations can be easily used to calculate the inclination angle ρ of the sliding failure surface, the weight W of the sliding block, the active earth pressure P_a, the passive earth pressure P_p, the active earth pressure coefficient K_a, and the passive earth pressure coefficient K_p for Coulomb's active and passive earth pressure conditions.

Then, governing equations for Coulomb's active and passive earth pressures were proved to have the same physical meanings as the graphical relationships obtained by satisfying $dP_a / d\rho = 0$ and $dP_p / d\rho = 0$ in Rebhann's drawing methods.

Based on the above research conclusions, the authors strongly suggest that when designing retaining walls in the future, for the convenience of calculation, the governing equations proposed in this paper for the Coulomb's active and passive earth pressures should be used directly. It is also suggested to incorporate the complete mathematical proofs provided in this paper into foundation engineering design textbooks and design specifications to enable the Coulomb's theory of active and passive earth pressures to be complete.

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